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Mixed-state twin observables

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Abstract. Twin observables are opposite subsystem observables A_+ and A_- if they are indistinguishable in measurement in a given mixed or pure state ρ in the sense that the measurement of one amounts to the same as the measurement of the other (direct measurement and so-called distant measurement, respectively). It is pointed out that twin observables may reveal quantum correlations that give rise to the disappearance of interference in the two-slit experiment. Twin observables in general states are investigated in detail algebraically and geometrically. It is shown that there is a far-reaching correspondence between the detectable (in ρ) spectral entities of the two operators. Twin observables are state-dependently quantum-logically equivalent, and the direct subsystem measurement of one of them *ipso facto* gives rise to the indirect (i.e. distant) measurement of the other. Existence of nontrivial twins requires a singularity of ρ. Systems in thermodynamic equilibrium do not admit subsystem twins. These observables may enable one to simplify the matrix representing ρ.

1. Introduction

Quantum correlations are one of the most peculiar and amazing physical ideas underlying quantum theory. They have been attracting much attention during the development of quantum mechanics. They are the corner stone for quantum measurement theory, which, applied to composite systems, makes transparent the conceptual background of quantum properties, revealing subtle interrelations between the subsystems [1]. The remarkable possibility of being able to predict a property of one of the subsystems on account of the result of the measurement performed on the other one, even when the subsystems are distant [2], is a substantial ingredient of modern theories of teleportation [3] and quantum computers [4]. Striking examples of strong or perfect correlations implying the disappearance of two-slit interference will be considered below (in the introduction). Most of the existing results apply to pure states of the composite system, reducing the scope of, and probably making harder, the potential applications. The main goal of this paper is to give several results extending the known properties of the pure state case [2, 5] to the general (i.e. mixed or pure) state, which is described by the statistical operator (density matrix).

In the pure state case the subsystem correlations are transparently sublimated in the notion of the twin observables. Let the quantum system S be composed of two subsystems S_+ and S_- (in short: $S = S_+ + S_-$); the corresponding state space being $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$, where $\mathcal{H}_s$ ($s = \pm$) are the state spaces of the subsystems. When S is prepared in a pure state $|\Phi\rangle \in \mathcal{H}$,

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then the partial traces of the projector $\rho = |\Phi\rangle\langle\Phi|$ (over the opposite subsystem spaces) give the statistical operators (mixtures of the second kind [6])

$$\rho_\pm \stackrel{\text{def}}{=} \mathrm{Tr}_\mp\, \rho. \tag{1}$$

Two opposite subsystem observables $\{A_+, A_-\}$ are called twin observables or twins if the measurement of A_s in $|\Phi\rangle$ amounts to the same as the measurement of A_{-s} in $|\Phi\rangle$. It turned out that for each s-subsystem observable A_s (acting in $\mathcal{H}_s$) compatible with ρ_s, there exists its twin (opposite subsystem) observable A_{-s}, compatible with ρ_{-s}. More precisely, both twin observables are uniquely defined only in the relevant subspaces $\mathcal{R}_\pm$, i.e. in the ranges of $\rho_\pm$. The subsystem states $\rho_\pm$ have equivalent spectral forms unless possible different defects (in particular, their ranges) are equi-dimensional. As a consequence, there exists (a $|\Phi\rangle$-dependent) anti-unitary mapping U_{a} of $\mathcal{R}_+$ onto $\mathcal{R}_-$ such that the restrictions of the twins to the corresponding relevant spaces $\mathcal{R}_\pm$ are U_{a} equivalent: $A_- = U_{\mathrm{a}} A_+ U_{\mathrm{a}}^\dagger$. Since the states in the null spaces of ρ_s are undetectable, this equivalence manifests itself as equivalence of the measurements of the twins with respect both to the obtained results (mean values) and to the post-measurement state (collapsed by the Lüders formula, [7]). In fact, the twins are indistinguishable by their action on the state $|\Phi\rangle$, $A_+|\Phi\rangle = A_-|\Phi\rangle$, or equivalently

$$A_+\rho = A_-\rho. \tag{2}$$

Hereafter A_+ stands also for the operator $A_+ \otimes \mathbf{1}_-$ in $\mathcal{H}$ ($\mathbf{1}_-$ is the identity in $\mathcal{H}_-$), etc.

Perfect correlations appear in a two-particle state when for each particle moving towards the upper or lower slit ($|\leftarrow u\rangle$, $|\leftarrow l\rangle$) another particle moves precisely in the opposite direction ($|\rightarrow u\rangle$, $|\rightarrow l\rangle$) [8]. The two-particle state vector with total linear momentum zero is

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|\leftarrow u\rangle_+|\rightarrow u\rangle_- + |\leftarrow l\rangle_+|\rightarrow l\rangle_-).$$

In an analogous single-particle composite state $|\Psi\rangle$ [9] the spatial states of the particle are $|u\rangle$, $|l\rangle$, denoting passage through the upper and the lower slit respectively, and the role of the second particle in the preceding example is played by linear (horizontal $|—\rangle$ or vertical $||\rangle$) polarization of the single particle:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|u\rangle_+||\rangle_- + |l\rangle_+|—\rangle_-).$$

This is, of course, achieved by putting corresponding polarizers in the slits.

Returning to the above two-particle state $|\Phi\rangle$, one has the following action of the correlation operator:

$$U_{\mathrm{a}}|\leftarrow u\rangle_+ = |\rightarrow u\rangle_- \qquad U_{\mathrm{a}}|\leftarrow l\rangle_+ = |\rightarrow l\rangle_-$$

and disappearance of interference due to the strong correlations between the two particles is obvious from the form of the statistical operator (1) of the first particle:

$$\rho_+ = \tfrac{1}{2}(|\leftarrow u\rangle_+\langle\leftarrow u|_+ + |\leftarrow l\rangle_+\langle\leftarrow l|_+).$$

In contrast to this, two-slit interference would stem from a one-particle state vector of the form $(|\leftarrow u\rangle_+ + |\leftarrow l\rangle_+)/\sqrt{2}$, i.e.,

$$\rho_+ = \tfrac{1}{2}(|\leftarrow u\rangle_+\langle\leftarrow u|_+ + |\leftarrow l\rangle_+\langle\leftarrow l|_+ + |\leftarrow u\rangle_+\langle\leftarrow l|_+ + |\leftarrow l\rangle_+\langle\leftarrow u|_+).$$

The correlation operator and the statistical operator of the spatial motion of the single-particle composite system $|\Psi\rangle$ act analogously, and lead analogously to the disappearance of interference.

The scope of definition (1) is quite general, i.e. it applies also to the mixed (composite) states, when the statistical operator ρ is not a projector. Analogously, the notions of the subsystem measurements and subsystem observables are by no means restricted to the pure states. However, in general, unlike in the pure state case, the subsystem states ρ_s are not simply related. In fact, the question of twins, i.e. mutually related opposite subsystem observables, seems to be completely open.

In section 2 we begin our investigation by analysing possible physically based ρ-dependent criteria for equivalence of a pair of opposite subsystem quantum events. Subsequently, we generalize them, regaining (2) as an unambiguous definition of twin observables. The importance of the null spaces is emphasized in section 3: when that of ρ is trivial only trivial twins exist, whereas the null space of the difference of the twins $(A_+ - A_-)$ contains the range of ρ. As is shown in section 4, indistinguishability in measurement has a physical content only for the reducees of the twins in the ranges of the subsystem states (1). Consequences of the specific spectral properties of twins are analysed in the next two sections.

We conclude this paper by a discussion of the physical meaning of twin observables within the framework of distant measurement. Although in the general case the mentioned anti-unitary mapping U_a does not exist, the equivalence of the twins in the measurement in ρ still holds true. As to practical motivation for this study, it is two fold.

(i) Perfect correlations between the physically relevant parts of twins will be established, and they are of interest when the ρ-imposed statistical connection between the two subsystems is of importance, e.g., in quantum information theory.
(ii) It turns out that the important problem of finding subsystem bases in which ρ is represented by a matrix that is as simple as possible can be solved with the help of twins.

2. Opposite subsystem equivalent events and twins

To generalize the notion of the twin observables for the given general, i.e. mixed or pure, quantum state (statistical operator or density matrix) ρ, we start with events (projection operators). At first we note that there are three seemingly different criteria (two of them physically based) which can be taken to define the equivalence of two events E and F in the state ρ.

Concerning their measurements, these events are observationally indistinguishable in ρ if their probabilities $\mathrm{Tr}\, E\rho$ and $\mathrm{Tr}\, F\rho$ are equal and the (unnormalized) states which result when the event occurs (gives result 1) in ideal measurements (i.e. the collapsed states [7]) are the same:

$$E\rho E = F\rho F. \tag{3}$$

Actually, this relation implies the equality of the probabilities.

In the special case we are interested in, when E and F are compatible (commuting) events with non-vanishing probabilities, equation (3) implies

$$\mathrm{Tr}\left(F\frac{E\rho E}{\mathrm{Tr}\, E\rho}\right) = \mathrm{Tr}\left(E\frac{F\rho F}{\mathrm{Tr}\, F\rho}\right) = 1. \tag{4}$$

This reveals the second criterion: the events E and F are state-dependently equivalent, $E \overset{\rho}{\sim} F$, each implying the other in ρ in a strengthened quantum-logical sense [10–12].

Notice that $E \overset{\rho}{\sim} F$ if and only if [11, remark 1] both the conditions $E^\perp FR = E^\perp F$, and $F^\perp R = EF^\perp$ are fulfilled (R being the null projector of ρ). Multiplying them by ρ from the right, one obtains $F\rho = EF\rho$ and $E\rho = EF\rho$, yielding the third, algebraic criterion

$$E\rho = F\rho. \tag{5}$$

Since (5) obviously implies (3), all these criteria are equivalent. They provide us with a satisfactory physical idea of equivalence or indistinguishability defining twin events. Relation (5) is very simple and hence it is preferable for further analysis (and it is a special case of (2)).

The physical meanings of (3) and (4) are objectionable because they are based on ideal measurements (underlying the Lüders collapse formula), which are almost unobtainable in the laboratory. Hence it is not worth exploring the equivalent expression (5) in its general form. However, when a two-subsystem composite system in the state ρ is considered, with $E = P_+$ and $F = P_-$ being opposite subsystem events, the mentioned objection may not be applied to (5), having the form:

$$P_+\rho = P_-\rho. \tag{6}$$

Indeed, any measurement (repeatable or unrepeatable) of P_s, not just its ideal measurement, if P_s occurs, leads [10, appendix A] to one and the same opposite subsystem conditional state $\mathrm{Tr}_s\, \rho P_s / \mathrm{Tr}\, P_s\rho$. In this case, the state-dependent implication does have empirical meaning and (6) is well worth investigating.

Having thus properly established (6), we now regain the starting relation (2) for twin observables. In fact, the underlying empirical idea of indistinguishability in measurement of the observables $A_\pm$ (in the state ρ) amounts now to analogous indistinguishability of all spectral measures (projectors) $P_\pm(\boldsymbol{B})$ of $A_\pm$ for arbitrary Borel subsets $\boldsymbol{B}$ of $\mathbb{R}$. Due to the equivalence of (3) and (5), we actually have $P_+(\boldsymbol{B})\rho = P_-(\boldsymbol{B})\rho$ for any Borel set $\boldsymbol{B}$. This, together with the well known functional dependence $A_\pm = \int_{\mathbb{R}} \lambda\, \mathrm{d}E_\lambda(A_\pm)$ (Stieltjes integral) of the observable on its spectral measure $E_\lambda(A_\pm) \equiv P_\pm(-\infty, \lambda]$ yields the claimed relation (2).

3. Role of the null spaces of ρ and $(A_+ - A_-)$

In the general state case of a composite system the compatibility of A_s with ρ_s does not guarantee the existence of its opposite subsystem twin A_{-s} as in the pure state case. In general, the problem 'Given the state ρ, what are all its twins?' is not easy to solve, and we do not try to do it in this article. Nevertheless, the presented study of twins gives quite general conditions for their existence.

The basic insight may be obtained when (2) is rewritten in the form

$$(A_+ - A_-)\rho = 0. \tag{7}$$

It immediately follows that the range of ρ is in the null space of $(A_+ - A_-)$. Hence, if ρ is nonsingular, A_+ and A_- must coincide, and then the only twins are trivial (equal scalar operators): $A_+ \otimes \mathbf{1}_- = \mathbf{1}_+ \otimes A_-$, implying $A_s = c\mathbf{1}_s$. More precisely, in a composite-system state ρ there exist nontrivial twin observables $A_\pm$ only if ρ is singular. It is noteworthy that this is analogous to the fact that the state-dependent equivalence $E \overset{\rho}{\sim} F$ can be nontrivial, i.e. not necessarily just $E = F$, if and only if ρ is singular. Thus, it is the possible nontrivial null space of ρ that gives the new quantum logical relations [10, corollary 2], as well as possible nontrivial twins.

Note further that (7) is a special ($a = 0$) case of the relation of the type $A\rho = a\rho$, with A Hermitian, ρ a statistical operator and a real. As shown in appendix A, this has a precise physical meaning: the measurement of the observable A in the state ρ gives the result a with certainty.

Let us return to the fact that (7) implies that a sufficient and necessary condition for the pair $A_\pm$ to be twins with respect to ρ is that the range of ρ is within the null space of $(A_+ - A_-)$. This has four immediate consequences.

C1: All the twins of ρ are also the twins of any state vector $|\Phi\rangle$ from the range $\mathcal{R}$ of ρ. In fact, the same is valid for the space $\overline{\mathcal{R}}$ topologically closing $\mathcal{R}$: the null space of $(A_+ - A_-)$, as every characteristic subspace, is topologically closed (because the Hermitian operators are closed even if they are unbounded), and therefore it contains $\overline{\mathcal{R}}$. (This claim is stronger than the preceding one if the range of ρ is infinitely dimensional and both twins are not bounded.)

C2: In any decomposition $\rho = \sum_i w_i |\Phi^{(i)}\rangle\langle\Phi^{(i)}|$ of ρ into pure states, at least all the twins of ρ itself are twins of each $|\Phi^{(i)}\rangle$. (This follows from C1.)

C3: The set of all ρ-twins is the intersection of the sets of all twins of the pure states contained in $\mathcal{R}$. Consequently, all composite states with the same range have the same twins; equivalently, this means that the ρ-twins are completely specified by the range of ρ only, and are not related to the finer information contended in the state.

C4: It may be interesting to get a criterion to recognize the states ρ admitting given opposite subsystem observables $A_\pm$ as twins. In fact, ρ is such a state if and only if it can be decomposed (as in C2 above) into pure states all being in the null space of $(A_+ - A_-)$. Moreover, such states can be decomposed into pure states in no other way.

Finally, (7) also shows that examples of twins may often come from additive observables. For instance, whenever the composite-system kinetic energy has the sharp value zero, the observable A_+ can be the kinetic energy of the first subsystem, while $-A_-$ is that of the second one, and analogously for linear momenta in the first example of the two-slit noninterference (in the introduction). In view of this, the conclusion C4 may be helpful in identification of the system state if such twins have been determined (by experimental evidence). Some other examples will be considered in section 6.

4. The detectable parts of the twins

We now begin the investigation of twin observables, i.e. of two Hermitian subsystem operators A_+ and A_- which satisfy relation (2). The subsystem state operators $\rho_\pm$ defined by (1) will play a role of paramount importance in our study, since they single out the subspaces essential for the twin observables concept. This is to a large extent due to their compatibility with twins (as in the pure state case)

$$[A_\pm, \rho_\pm] = 0. \tag{8}$$

In fact, using elementary identities for the partial traces and subsystem operators, and (in the last but one equality) the adjoint of (2), one directly verifies

$$A_\pm \operatorname{Tr}_\mp \rho = \operatorname{Tr}_\mp A_\pm \rho = \operatorname{Tr}_\mp A_\mp \rho = \operatorname{Tr}_\mp \rho A_\mp = \operatorname{Tr}_\mp \rho A_\pm = (\operatorname{Tr}_\mp \rho) A_\pm.$$

It is an immediate consequence that the subsystem Hermitian operator A_s commutes also with all characteristic projectors of the corresponding subsystem state operator ρ_s, and therefore with any sum of them. In particular, the sum of all characteristic projectors with positive corresponding characteristic values is the range projector R_s of ρ_s and the zero characteristic value projector is the null space projector N_s. Therefore,

$$[A_\pm, R_\pm] = 0 \qquad [A_\pm, N_\pm] = 0. \tag{9}$$

These relations imply that the twins reduce in the range $\mathcal{R}_s$ and the null space $\mathcal{N}_s$ of the corresponding subsystem state operator. We denote by A'_s and A''_s the reducees in $\mathcal{R}_s$ and $\mathcal{N}_s$ respectively, and we call them the detectable and the undetectable parts of A_s, because only the detectable parts influence measurements (as will be argued in section 5). The decompositions

$$A_\pm = A'_\pm \oplus A''_\pm \equiv (A'_\pm \oplus \mathbf{0}''_\pm) + (\mathbf{0}'_\pm \oplus A''_\pm) \tag{10}$$

thus obtained are, of course, paralleling the decompositions $\mathcal{H}_\pm = \mathcal{R}_\pm \oplus \mathcal{N}_\pm$ of the subsystem spaces (**0** stands for the null operator in the corresponding subspace).

As shown in appendix B, there is a relation among the system and subsystem state ranges (being accompanied by the equivalent projector relations):

$$\mathcal{R} \subseteq (\mathcal{R}_+ \otimes \mathcal{R}_-) \qquad R = RR_+R_- \tag{11}$$

entailing

$$(\mathcal{R}_+ \otimes \mathcal{H}_-) \supseteq \mathcal{R} \subseteq (\mathcal{H}_+ \otimes \mathcal{R}_-) \qquad R = RR_\pm = R_\pm R. \tag{12a}$$

Hence,

$$\rho = R\rho = R_\pm R\rho = R_\pm \rho$$

showing that the range projectors of the subsystem state operators are always twins. Note that in the pure state case the subsystem state operators $\rho_\pm$ themselves are always twins [2, equation (31) and theorem 8].

Taking the orthocomplements of (12a), one can write the following relations for the null spaces $\mathcal{N}_\pm$, $\mathcal{N}$ of the (sub)systems states, and the corresponding projectors $N_\pm$, N:

$$(\mathcal{N}_+ \otimes \mathcal{H}_-) \subseteq \mathcal{N} \supseteq (\mathcal{H}_+ \otimes \mathcal{N}_-) \qquad N_\pm N = N_\pm. \tag{12b}$$

Since $N\rho = 0$ this yields $N_\pm \rho = 0$. Taking into account that $\mathbf{0}'_\pm \oplus A''_\pm = (\mathbf{0}'_\pm \oplus A''_\pm)N_\pm$, this entails

$$(\mathbf{0}'_+ \oplus A''_+)\rho = (\mathbf{0}'_- \oplus A''_-)\rho = 0 \tag{13}$$

i.e., the undetectable components are twins (in a trivial way). Replacing the decompositions (10) in (2), and taking into account the last relation, one obtains

$$(A'_+ \oplus \mathbf{0}''_+)\rho = (A'_- \oplus \mathbf{0}''_-)\rho \tag{14}$$

i.e. the detectable components of twins are also, in their turn, twins.

Henceforth, the prime on a subsystem entity will denote its restriction to the subspace $\mathcal{R}_s$, and the double prime the restriction on $\mathcal{N}_s$. On the other hand, the composite-system entities attain prime and double prime restrictions when restricted to the subspace $(\mathcal{R}_+ \otimes \mathcal{R}_-)$ and its orthocomplement, respectively.

One can strengthen the detectable–undetectable aspect of twins as follows. Two opposite subsystem Hermitian operators A_+ and A_- are twins if and only if the following two conditions are satisfied:

(i) commutation $[A_\pm, R_\pm] = 0$; and
(ii) their detectable parts $A'_\pm$ are twins for ρ'.

That these conditions are not only necessary, but also sufficient is, first of all, obvious from the fact that (i) amounts to the same as the existence of the two parts of each operator $A_\pm$. Further, the undetectable parts, whatever they are, are always twins for ρ'' because $\rho'' = 0$ (as ρ is restricted to a part of its null space).

From the analytical point of view, it is not practical to keep the null spaces of the three state operators included (because everything of interest is trivially zero in them). Henceforth we mostly restrict the state space $\mathcal{H}_+ \otimes \mathcal{H}_-$ to its subspace $\mathcal{R}_+ \otimes \mathcal{R}_-$. The twin relation (2) now reduces to the effective part of (14):

$$A'_+\rho' = A'_-\rho'.$$

In accordance with the remark after (7), if ρ' is nonsingular, then there are no nontrivial twins (in $\mathcal{R}_+ \otimes \mathcal{R}_-$). However, if simultaneously the nonreduced state operator ρ itself is singular, then it does have nontrivial twins, but these are $\alpha R_\pm$, $\alpha \in \mathbb{R}$. (We have utilized the obvious fact that if $\{A_+, A_-\}$ are twins, so are $\{\alpha A_+, \alpha A_-\}$, $\alpha \in \mathbb{R}$.)

5. Characteristic vectors and the characteristic values of the twins

Considering the commutation relations (8), one infers that the twin operators reduce in each characteristic subspace of the corresponding state operator, and these, except the null space, are necessarily finite dimensional (because the positive characteristic values have to add up, repetitions included, to 1). As a consequence, the spectra of the twin operators have to be purely discrete (in $\mathcal{R}_+ \otimes \mathcal{R}_-$).

For further study, we confine ourselves to the subspace $\mathcal{R}_+ \otimes \mathcal{R}_-$. Let us take arbitrary characteristic orthonormal (ON) bases for the given twins $A'_\pm$ in the given composite-system state ρ'. We denote the bases by $\{|m_\pm\rangle|\forall m_\pm\}$, and the corresponding characteristic values by $a^\pm_{m_\pm}$. One obtains the pair of the characteristic equations

$$A'_\pm|m_+\rangle|m_-\rangle = a^\pm_{m_\pm}|m_+\rangle|m_-\rangle.$$

When ρ' is applied (from the left) to both equations, and the results are subtracted, the adjoint of (7) gives

$$0 = (a^+_{m_+} - a^-_{m_-})\rho'|m_+\rangle|m_-\rangle.$$

If we assume that $a^+_{m_+} \neq a^-_{m_-}$, then $\rho'|m_+\rangle|m_-\rangle = 0$, i.e.

$$|m_+\rangle|m_-\rangle \in \mathcal{N}' \qquad \text{if} \quad a^+_{m_+} \neq a^-_{m_-}. \tag{15}$$

Evidently, representing ρ' in these bases may give a substantial simplification of the matrix if the twins (or a set of mutually commuting twins) are chosen with degeneracies of the characteristic values a_n as small as possible. (See section 6 for the best possible case of simplification in this way.)

To derive a consequence of (15), let us assume that a^+ belongs to the spectrum of A'_+, but not to that of A'_-. If $|a^+\rangle$ is a corresponding characteristic vector, then on account of (15) one obtains for an arbitrary $|\psi\rangle \in \mathcal{R}_+$,

$$\langle\psi|\rho'_+|a^+\rangle = \sum_{m_-}\langle\psi|\langle m_-|\rho'|a^+\rangle|m_-\rangle = 0.$$

Since $|\psi\rangle$ is arbitrary, we end up with $\rho'_+|a^+\rangle = 0$, which contradicts our starting assumption. The symmetric argument follows this analogously.

Thus, we conclude that the twins $A'_\pm$ must have equal spectra

$$\sigma' = \sigma'(A_\pm) = \sigma(A'_\pm). \tag{16}$$

Notice that in general the corresponding multiplicities are not equal, as clearly illustrated by the example of $cR_\pm$ above, despite the particular equality in the pure state case [2, 5].

In order to simplify the forthcoming study of the characteristic projectors, we prove in appendix C the following basic structural properties of the set of all the twins for a given composite state.

(i) For an arbitrary (Hermitian) operator function F on the twins $A_\pm$, the operators $F(A_+)$ and $F(A_-)$ are also twins, i.e.

$$F(A_+)\rho = F(A_-)\rho. \tag{17a}$$

(ii) The set of pairs of ρ-twin observables is a symmetric polynomial algebra, i.e. any (real) symmetric polynomial $F(x, y, \ldots)$ maps pairs $A_\pm, B_\pm, \ldots$ of twin observables into twins:

$$F(A_+, B_+, \ldots)\rho = F(A_-, B_-, \ldots)\rho. \tag{17b}$$

We emphasize the consequence that (Hermitian) twin observables form real vector spaces, enabling us to define the set by its basis. These results are also helpful when new twins are to be generated from known ones.

Now we turn to the characteristic projectors $P_\pm(a^\pm)$ corresponding to the characteristic values $a^\pm$ of the twins $A_\pm$. Note that (9) implies $[P_\pm(a^\pm), R_\pm] = 0$ for any $a^\pm$. Then the equality

$$\mathrm{Tr}\, P_\pm(a^\pm)\rho = \mathrm{Tr}\, P'_\pm(a^\pm)\rho' + \mathrm{Tr}\, P''_\pm(a^\pm)\rho'' = \mathrm{Tr}\, P'_\pm(a^\pm)\rho'$$

reveals that the positive probability characteristic values of twins $A_\pm$ are those and only those remaining in the spectra of $A'_\pm$. This means physically that only the spectral events of $A'_\pm$ are detectable in ρ. This justifies the term 'detectable part'.

Moreover, due to the equal spectra of the detectable parts (cf (16)), the well known spectral projector polynomials are the same function for both twins:

$$P'_\pm(a) = \prod_{(a\neq)b\in\sigma'} \frac{A'_\pm - b}{a - b} \qquad \forall a \in \sigma'.$$

In view of (17*a*), $P'_+(a)\rho' = P'_-(a)\rho'$ follows. To conclude, all positive probability characteristic projectors $P'_\pm(a)$ of twins are twins. This is not only necessary but also sufficient, because

$$A'_\pm\rho' = \sum_{a\in\sigma'} a P'_\pm(a)\rho'.$$

On account of (13), $P_\pm(a)$, $a \in \sigma'$ are also twins, and

$$A_\pm\rho = \sum_{a\in\sigma'} a P_\pm(a)\rho.$$

As to the undetectable parts $\sigma''(A_\pm)$ of the spectra, each possible pair of the characteristic projectors are also twins, but annihilate ρ, in the trivial sense of (13).

6. Complete twins and examples

Let ρ'_{12} be such that a pair of twins $A'_\pm$ with all characteristic values nondegenerate exists. We call such operators complete twins. Since now the spectra of the two operators completely coincide, i.e. the multiplicities are also equal, their ranges $\mathcal{R}_\pm$ are equi-dimensional. This is a necessary condition for the existence of complete twins.

If ρ' does have a pair of complete twins $A'_\pm$ with the common spectrum σ' and corresponding characteristic ON bases $\{|a\rangle_\pm \mid a \in \sigma'\}$, then due to (15) the matrix representing ρ' has the form

$$\langle a|\langle c|\rho'|b\rangle|d\rangle = \delta_{ac}\delta_{bd}\langle a|\langle a|\rho'|b\rangle|b\rangle. \tag{18}$$

This is the maximal simplification that one can achieve by using a pair of twins or a pair of sets of mutually compatible twins.

As mentioned in the introduction, when the composite-system state is pure, $\rho' \equiv |\Phi\rangle\langle\Phi|$, each subsystem observable compatible with the corresponding subsystem state has its opposite subsystem twin. Therefore, complete twins now exist [2, 5]. The vectors in a common characteristic bases of $\rho'_\pm$ and such an operator $A'_\pm$, $\{|a\rangle_\pm|a \in \sigma'\}$ (the corresponding characteristic values r_a of $\rho_\pm$ are the same) are determined up to phase factors. If these are simultaneously chosen in both subsystem spaces according to

$$|a\rangle_- = U_\mathrm{a}|a\rangle_+ \stackrel{\mathrm{def}}{=} \rho'^{-1/2}_-\langle a|_+|\Phi\rangle \qquad \forall a \in \sigma' \tag{19a}$$

(a partial scalar product on the right), then such bases give the Schmidt canonical forms (or biorthogonal expansions) of $|\Phi\rangle$:

$$|\Phi\rangle = \sum_a r_a^{1/2} |a\rangle|a\rangle. \qquad (19b)$$

Let

$$\rho' = \sum_i w_i |\Phi^{(i)}\rangle\langle\Phi^{(i)}|$$

be a decomposition of the given composite-system state ρ' into pure states (orthogonal or not), and let the state ρ' have a pair of complete twins $A'_\pm$. Then, according to the conclusion C2 of section 3, this is a common pair of twins for all admixed states $|\Phi^{(i)}\rangle\langle\Phi^{(i)}|$. It will not necessarily lead to simultaneous Schmidt canonical forms (19*b*) because the conditions (19*a*) need not be simultaneously satisfied. However, if one relaxes the requirement that the biorthogonal expansion has positive expansion coefficients, then, with an arbitrary pair of characteristic bases of the twins, one obtains simultaneous generalized Schmidt biorthogonal expansions:

$$|\Phi^{(i)}\rangle = \sum_a \alpha_a^{(i)} |a\rangle|a\rangle \qquad \forall i \quad (\alpha_a^{(i)} \in \mathbb{C}). \qquad (20)$$

The bases $|a\rangle_\pm$ are characteristic ones, not only for $\rho_\pm$, but also for all pure state subsystem operators $\rho_\pm^{(i)} = \mathrm{Tr}_\mp |\Phi^{(i)}\rangle\langle\Phi^{(i)}|$ (the corresponding characteristic values are $r_a^{(i)} = |\alpha_a^{(i)}|^2$). Consequently, having these characteristic bases simultaneously, all operators

$$A_s, \{\rho_s^{(i)} : \forall i\}, \rho_s \qquad (s = \pm) \qquad (21)$$

are compatible (separately for each s of course).

A mixed state ρ' with a pair of complete twins can be obtained by mixing pure states that do have common generalized Schmidt biorthogonal expansions (20). Then, any other decomposition of the composite-system state (the spectral form included), also has this property.

To conclude the section, we give several examples of complete twins in the two-particle spin spaces. The observable of the single-particle z-component of spin is denoted by $s_{z\pm}$, yielding quantum numbers $m_\pm$, while S and M_S are the total spin and its z-projection quantum numbers.

Example 1. The state space of the system of two spin-$\frac{1}{2}$ particles is $\mathbb{C}^2 \otimes \mathbb{C}^2$. When the range of a mixed state ρ is spanned by the two state vectors $|S = 1, M_S = 0\rangle$ and $|S = 0, M_S = 0\rangle$, all twins have the form $A_\pm = \alpha \mathbf{1}_\pm \pm \beta s_{z\pm}$ $(\alpha, \beta \in \mathbb{R})$. Thus, the additive-type complete twins $\pm s_{z\pm}$, together with (trivial) identities, span the two-dimensional twin space. Let us note that the (at first sight) similar example of the states with the range spanned by $|S = 1, M_S = 0\rangle$ and $|S = 1, M_S = -1\rangle$, admitting only $\alpha \mathbf{1}_\pm$ twins, illustrates that singularity of ρ is not sufficient for the occurrence of nontrivial twins.

Example 2. For two spin-1 particles the state space is $\mathbb{C}^3 \otimes \mathbb{C}^3$. Whenever $\mathcal{R}$ is the entire $M_S = 0$ subspace spanned by the three vectors $|S = 2, M_S = 0\rangle$, $|S = 1, M_S = 0\rangle$ and $|S = 0, M_S = 0\rangle$, the space of the twins is spanned by $\mathbf{1}_\pm$, $s_{z\pm}^2$ and $\pm s_{z\pm}$. Only the last one is a complete twin, although also the second one is nontrivial. A more subtle analysis may be performed for the states with the range $\mathcal{R}$ equal to the $M_S = 1$ subspace, spanned by $|S = 2, M_S = 1\rangle$ and $|S = 1, M_S = 1\rangle$. Here the single-particle state operators $\rho_\pm$ are singular (with the null spaces spanned by $m_\pm = -1$ states), in contrast to the former examples. In fact, the twin space is spanned by $\mathbf{1}_\pm$, $A_\pm = \pm s_{z\pm} \mp \frac{1}{2}\mathbf{1}_\pm$ and $s_{z\pm}^2 - s_{z\pm}$. In addition to the first pair, the last one is also trivial, but in the sense of (13) as a pair of undetectable observables. On the other hand, $A_\pm$ are complete twins, based on the sharply valued additive observable S_z (on $\mathcal{R}$) (cf the final remark in the next section). Their detectable parts $A'_\pm = \pm\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are complete twins; supplemented by $\mathbf{0}''_\pm$ in the undetectable null spaces these yield the twins $A'_\pm \oplus \mathbf{0}''_\pm = \mp\frac{1}{2}\mathbf{1}_\pm \pm \frac{3}{4}s^2_{z\pm} \pm \frac{1}{4}s_{z\pm}$ which, together with the subsystem range projectors $R_\pm = \mathbf{1}_\pm - \frac{1}{2}s^2_{z\pm} + \frac{1}{2}s_{z\pm}$, span the space of the detectable twins.

7. Concluding remarks

Examples of strong or perfect correlations between two subsystems in a quantum state were given in the introduction. There are many other known cases in the literature, and their importance for quantum communication theory is well understood. Twin observables for pure or mixed states are a concrete physical realization of strong correlations.

The fact that the detectable parts $A'_\pm$ of twin observables in a composite-system state ρ have common discrete characteristic values $a \in \sigma'$ with possibly differing multiplicities and no continuous spectrum has various consequences.

First of all, the physical meaning of twin observables becomes transparent in terms of distant measurement, when the experimental indistinguishability of A_+ and A_- in ρ is expressed in a more detailed way as follows. Any result a is obtained with the same probability irrespectively if A_+ or A_- is measured in ρ, and, if the measurement is ideal, it is accompanied by the same change of state:

$$\rho \mapsto \frac{P_+(a)\rho P_+(a)}{\mathrm{Tr}\ P_+(a)\rho} = \frac{P_-(a)\rho P_-(a)}{\mathrm{Tr}\ P_-(a)\rho}. \tag{22}$$

Naturally, the expectation values $\langle A_+\rangle$ and $\langle A_-\rangle$ are also the same in ρ.

This fact is, perhaps, more intriguing when put in the following way. One can measure A_- indirectly, or, as one says, distantly, in ρ as a sheer consequence of the actual direct measurement of the nearby opposite subsystem observable A_+ in this state (or vice versa). When two particles in a correlated state are literally distant from each other, then this measurement of the distant second-particle observable is performed in a ghostly way, by not 'touching' dynamically the distant particle, and performing measurements only on the nearby (first) particle.

The ρ-dependent (strengthened) quantum-logical equivalence of $P_+(a)$ and $P_-(a)$ for a mentioned result a can be spelled out as follows. If $P_+(a)$ occurs in an arbitrary measurement, the opposite subsystem event $P_-(a)$ becomes certain, and vice versa. Since this claim does not involve the global behaviour of the composite system, we can refer to this property of $P_\pm(a)$ as local physical twins. In contrast, having in mind the common global change of state (22), we can speak of these events as global physical twins.

It turns out that the correlations between the subsystems are closely connected with the null space of the composite state. Indeed, physically nontrivial twins exist only for singular ρ; otherwise, the only twins are $\alpha\mathbf{1}_\pm$ ($\alpha \in \mathbb{R}$), when their measurements are without content. Consequently, a complex system in equilibrium that is described by the canonical ensemble state $\rho = \mathrm{e}^{-H/kT}/\,\mathrm{Tr}\,\mathrm{e}^{-H/kT}$ cannot be decomposed into twin-rigged subsystems for $T > 0$ (when ρ is nonsingular). In fact, the conclusion C3 in section 3 shows that only geometrical relations among the (sub)system state ranges $\mathcal{R}_\pm$ and $\mathcal{R}$ are relevant for the twin-type correlations.

Nevertheless, the problem of determining the set of all twins for a given ρ remains hard enough. (The direct solution of the linear system (2) is possible only for especially simple, thus mostly unrealistic, systems.) The structural properties stated by (17) should be helpful.

On the other hand, one should note that the additive observables are promising candidates for twin observables. In fact, whenever a composite system is prepared in a state ρ with the sharp value, say b, of an additive observable $B = B_+ + B_-$, then the subsystem observables $\pm B_\pm \mp \frac{b}{2}\mathbf{1}_\pm$ are twins as follows from (7). Note that the range $\mathcal{R}$ is in this case confined to the corresponding characteristic subspace of B, i.e. $RP(b) = R$, in accordance with the required singularity of ρ.

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Appendix A. Certainty of a in a state satisfying $A\rho = a\rho$

Applying the relation $A\rho = a\rho$ to an arbitrary state vector $|\psi\rangle$ one obtains $A\rho|\psi\rangle = a\rho|\psi\rangle$. This means that $\rho|\psi\rangle$ is a characteristic vector of A for the characteristic value a, independent of $|\psi\rangle$, i.e. the range of ρ is within range of the corresponding characteristic projector P_a of A. Therefore, $P_a\rho = \rho$, implying that the probability Tr $P_a\rho$ of the result a measuring A in ρ is 1.

Moreover, the condition we have started with is also necessary for the result a in the measurement of A in ρ to be certain, because the entire argument can be read backwards. The only nontrivial step, i.e., that Tr $P_a\rho = 1$ implies $P_a\rho = \rho$, is proven in [10, lemma A.2]. Thus, this probability-one statement is the precise physical meaning of the condition at issue.

Appendix B. Relation among the ranges and null spaces

For arbitrary normalized $|\psi_-\rangle \in \mathcal{H}_-$ there is an ON basis $\{|i_-\rangle|\forall i_-\}$ of $\mathcal{H}_-$ containing $|\psi_-\rangle$. Then, since ρ is positive, the definition (1) gives for each $|n_+\rangle$ from the null space $\mathcal{N}_+$ of ρ_+

$$0 \leqslant \langle n_+|\langle\psi_-|\rho|n_+\rangle|\psi_-\rangle \leqslant \sum_{i_-}\langle n_+|\langle i_-|\rho|n_+\rangle|i_-\rangle = \langle n_+|\rho_+|n_+\rangle = 0.$$

Thus, $|n_+\rangle|\psi_-\rangle$ is in the null space $\mathcal{N}$ of ρ, i.e. $(\mathcal{N}_+\otimes\mathcal{H}_-) \subseteq \mathcal{N}$; symmetrically: $(\mathcal{H}_+\otimes\mathcal{N}_-) \subseteq \mathcal{N}$. In addition, $\mathcal{H} = (\mathcal{N}_+\otimes\mathcal{N}_-)\oplus(\mathcal{N}_+\otimes\mathcal{R}_-) \oplus (\mathcal{R}_+\otimes\mathcal{N}_-)\oplus(\mathcal{R}_+\otimes\mathcal{R}_-)$, since the ranges $\mathcal{R}_\pm$ and $\mathcal{R}$ of the (sub)system states are orthocomplements of $\mathcal{N}_\pm$ and $\mathcal{N}$ respectively. Finally, it follows

$$\mathcal{N} \supseteq (\mathcal{N}_+\otimes\mathcal{N}_-)\oplus(\mathcal{N}_+\otimes\mathcal{R}_-)\oplus(\mathcal{R}_+\otimes\mathcal{N}_-) \tag{B.1}$$

directly implying (11).

Appendix C. Proof of structural properties (17) of the set of twins

To prove (17a), let $\{|m_\pm\rangle|\forall m_\pm\}$ be characteristic complete ON bases of the respective range projectors $R_\pm$ (of $\rho_\pm$), such that their sub-bases are also characteristic bases of the respective twins $A'_\pm$. Then we verify the twin relation in the form of the adjoint of (7) for $F(A_\pm)$. Indeed,

$$\rho(F(A_+) - F(A_-))|m_+\rangle|m_-\rangle = (F(a^+_{m_+}) - F(a^-_{m_-}))\rho|m_+\rangle|m_-\rangle = 0$$

since $|m_+\rangle|m_-\rangle \in \mathcal{N}$ whenever $|m_+\rangle \in \mathcal{N}_+$ and/or $|m_-\rangle \in \mathcal{N}_-$ (by (12b)), or $a^+_{m_+} \neq a^-_{m_-}$ (by (15)), and in the remaining case (i.e. when $|m_\pm\rangle \in \mathcal{R}_\pm$ and $a^+_{m_+} = a^-_{m_-}$) obviously $F(a^+_{m_+}) - F(a^-_{m_-}) = 0$.

As for (17*b*), a straightforward consequence of (2) is that $\alpha A_\pm + \beta B_\pm$ are twins, while the assertion on the symmetric products follows from

$$(A_+B_+ + B_+A_+)\rho = (A_+B_- + B_+A_-)\rho = (B_-A_+ + A_-B_+)\rho = (B_-A_- + A_-B_-)\rho.$$

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